Midterm – Numerical Simulation of Turbulence

All problems are weighted equally in the calculation of the grade.

1. The evolution equation for the kinetic energy $\hat{E}(\kappa, t)$ in Fourier mode κ is given by

$$\frac{d}{dt}\hat{E}(\kappa, t) = \hat{T}(\kappa, t) - 2\nu\kappa^2\hat{E}(\kappa, t),$$

where ν is the viscosity of the fluid.

   (a) Explain the two components in the above right-hand side in terms of their physical meaning.

   (b) Illustrate the idea of the energy cascade in turbulence with reference to the above equation.

   (c) Kolmogorov's 1941 (K41) turbulence theory is based on two core assumptions. Formulate these assumptions.

2. In large-eddy simulation (LES), a spatial filter is introduced. In this case, we focus on a convolution filter applied in a single spatial dimension. The filtered velocity is then given by

$$\overline{u}(x, t) = \int_{-\infty}^{\infty} G(x - \xi)u(\xi, t)d\xi,$$

where u denotes the velocity and the convolution kernel satifies $\int_{-\infty}^{\infty} G(x)dx = 1$, $G(-x) = G(x)$ and $G(x) \to 0$ as $x \to \pm\infty$

   (a) Show that

$$\frac{\partial \overline{u}}{\partial x} = \overline{\frac{\partial u}{\partial x}}$$

   (b) Determine a Taylor expansion of $u(\xi, t)$ around (x, t).

   (c) Use (b) to show that

$$\overline{u}(x, t) \approx u(x, t) - \alpha\frac{\partial^2 u}{\partial x^2}$$

   and express the coefficient α in terms of G.

3. The velocity field $u(x, t)$ satisfies the incompressible Navier–Stokes equations. A spatial filter is applied to this field, yielding the filtered velocity $\overline{u}$. The correspon-ding fluctuation, or residual component, is defined as $u' = u - \overline{u}$.

   (a) State the (incompressible) Navier-Stokes equations.

   (b) Derive the LES equations for the filtered velocity $\overline{u}$ from (a).

(c) Outline the closure problem in LES.

4. Consider turbulent channel flow in the usual coordinate system, so that y is the normal distance from the wall. Very close to the wall, to leading order in y, the velocity field can be written

$$u_1(x,y,z,t) = a(x,z,t)y, \quad u_2(x,y,z,t) = b(x,z,t)y^2, \quad u_3(x,y,z,t) = c(x,z,t)y.$$

where a, b, and c are arbitrary functions. Filtering is applied in x–z planes only, using a uniform filter of width Δ.

(a) Show that the filtered velocity field is

$$\overline{u}_1 = \overline{a}y, \quad \overline{u}_2 = \overline{b}y^2, \quad \overline{u}_3 = \overline{c}y.$$

(b) Determine the residual stresses τ_{11}^R, τ_{22}^R and τ_{21}^R